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Kubo–Mori–Bogoliubov Fisher information on the quantum Gaussian model and violation of the scale invariance

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Abstract
The classical Gaussian model, a parametric family of the Gaussian distribution, is known to be a space of constant negative curvature if one regards the Fisher information on the model as a Riemannian metric. Constant curvature reflects the scale invariance of the classical Gaussian model, which is well known in information geometry. However, it is shown that if the Kubo–Mori–Bogoliubov Fisher information on the quantum Gaussian model is adopted as a Riemannian metric, then this scale invariance on the Gaussian model is broken due to the quantum effect. In the present study, the connection between the geometry of the classical Gaussian model and its quantum counterpart is clarified using the Taylor expansion with respect to the Planck constant. It is further shown that such a method is not applicable to a finite-dimensional system such as the spin system.

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1. Introduction

In classical parameter estimation, a parametric family of probability distributions is assumed

$$\mathcal{M}_{\mathrm{cl}} := \{p_\theta(x) : \theta \in \Theta \subset \mathbb{R}^k\}, \qquad (1)$$

and the unknown parameter is inferred from sample data $x_1, x_2, \ldots$. The unknown parameter $\theta = (\theta^1, \ldots, \theta^k)$ of a probability density $p_\theta(x) \in \mathcal{M}_{\mathrm{cl}}$ is considered as a coordinate, and the family (1) is referred to herein as a *(classical) model manifold*, or as simply as a *model* [1].

Čencov showed that there exists an essentially unique Riemannian metric up to a constant factor on a model if monotonicity is imposed under stochastic maps [4]. Thus, a model can be regarded as a Riemannian manifold. In particular, this metric is referred to as the *Fisher information*, or *Fisher metric*, and is derived as

$$g_{ij}^{(\mathrm{cl})}(\theta) = \frac{\partial^2}{\partial\theta^j\partial\theta'^i} D(p_{\theta'}\|p_\theta)\bigg|_{\theta'\to\theta}, \tag{2}$$

where $D(p\|q) := \int p(x)\log\left(\frac{p(x)}{q(x)}\right)\mathrm{d}x$ is the *Kullback–Leibler divergence* or the *relative entropy*, which in classical statistics is considered to be a natural distance-like measure from $p(x)$ to $q(x)$ in the totality of all probability densities. The inverse matrix $(g^{(\mathrm{cl})})^{ij}$ gives the limitation of the accuracy of the parameter estimation in the corresponding family of the probability density. Roughly speaking, the Fisher information represents the extent to which one can distinguish between $p_{\theta+d\theta}$ and p_θ from sample data.

Next, let us consider the quantum setting. Let $\mathcal{H}$ be a separable (possibly infinite-dimensional) Hilbert space of a quantum system. The unknown quantum state is described by a positive operator of trace one on $\mathcal{H}$. This operator is referred to as a *state* or a *density operator* and is a quantum analogue of probability distributions. Let $\mathcal{S}(\mathcal{H})$ denote the set of all density operators on $\mathcal{H}$, i.e., $\mathcal{S}(\mathcal{H}) := \{\rho : \mathrm{Tr}\,\rho = 1, \rho \geqslant 0\}$. A parametric family of density operators,

$$\mathcal{M} := \{\rho_\theta \in \mathcal{S}(\mathcal{H}) : \theta \in \Theta \subset \mathbb{R}^k\},$$

is referred to as a *(quantum) model* and a quantum analogue of a classical model. In addition, quantum relative entropy [22], a quantum version of the Kullback–Leibler divergence, is defined as follows:

$$D(\rho\|\sigma) := \mathrm{Tr}[\rho(\log\rho - \log\sigma)], \qquad \rho, \sigma \in \mathcal{S}(\mathcal{H}), \tag{3}$$

and plays a crucial role in several topics of quantum information, for instance, quantum channel coding [13, 25] and quantum hypothesis testing [10, 21]. Thus, it seems natural to adopt the second derivative of (3) as a quantum analogue of the Fisher information (2). A formal definition based on the Kubo–Mori–Bogoliubov (KMB) inner product [18] (or canonical correlation in linear response theory) is not given here, and KMB Fisher information is derived only from the quantum relative entropy parallel to (2),

$$g_{ij}^{\mathrm{KMB}}(\theta) = \frac{\partial^2}{\partial\theta^j\partial\theta'^i} D(\rho_{\theta'}\|\rho_\theta)\bigg|_{\theta'\to\theta}. \tag{4}$$

The above formula is sufficient for the purposes of the present study. KMB Fisher information is not a unique extension of the classical Fisher information. Petz [23, 24] showed that there exist many Riemannian metrics on a quantum model manifold under the same condition as Čencov imposed on a classical model. This arbitrariness is due to noncommutativity.

Thus, various extensions have been developed. For example, the Fisher information based on the symmetric logarithmic derivative (SLD Fisher information) was proposed by Helstrom [9]. SLD Fisher information gives the most informative Cramér–Rao bound in the one-parameter unbiased estimation [12, 20]. The Fisher information based on the right logarithmic derivative (RLD Fisher information) is another candidate. RLD Fisher information gives the best bound in a certain two-parameter estimation [12, 30]. According to Hayashi [7], with respect to large deviation, SLD Fisher information is preferable to KMB Fisher information. At present, KMB Fisher information has not yet been given such characterization with regard to practical application.

From the above argument, the question may arise as to whether KMB Fisher information is significant in practical application. However, as described below, there is evidence of the significance of KMB Fisher information.

In classical Bayesian prediction, when we have no knowledge on the unknown parameter θ specifying $p_\theta(x)$, the volume element of the model manifold (i.e., $\sqrt{\det g_{ij}^{(\mathrm{cl})}(\theta)}$) is often adopted as a noninformative prior distribution of θ, and the prediction method based on this prior distribution is used. However, there exists another noninformative prior distribution that gives a better prediction method when the Fisher information satisfies a certain geometrical condition. Detailed explanation is not the central issue of the present paper and omitted. (See, Komaki [17]) But the key to such arguments is that the Fisher information is obtained by differentiating the Kullback–Leibler divergence (the classical relative entropy) as indicated by (2).

In the quantum Bayesian prediction, which was recently formulated by Tanaka and Komaki [27], the analogous argument is expected to hold with some modification. Although there are several kinds of the Fisher information on the quantum statistical models, at least in quantum Bayesian prediction, it seems the most natural to focus on the Fisher information derived from the quantum relative entropy, i.e., KMB Fisher information, because KMB Fisher information is that obtained by differentiating the quantum relative entropy as indicated by (4).

Further, in quantum information geometry, KMB Fisher information has a distinguishing feature. Among all of the quantum Fisher information with desirable properties, only KMB Fisher information allows a quantum model manifold to have a torsion-free affine connection [1].

As mentioned above, we expect that KMB Fisher information plays a crucial role in some application and thus, investigation of the geometrical structure on a specific model endowed with KMB Fisher information is significant not only from mathematical viewpoint but also from practical viewpoint. For finite-dimensional systems, such as the spin system, a number of studies [2, 5, 11, 19, 23] have investigated the geometrical structure on quantum model manifolds endowed with KMB Fisher information. However, few studies have examined the infinite-dimensional system [14, 15]. Thus, the present study investigates the geometrical structure on the quantum Gaussian model endowed with KMB Fisher information. The scalar curvature is found to be negative. This result is similar to that of the classical counterpart. While the classical Gaussian model is treated as a typical example, both theoretically and practically, the quantum model has not been considered as such. In the present paper, it is clarified how these two models are connected with each other. Particular emphasis is placed on the fact that the change of geometrical structure in the classical Gaussian model can be interpreted as being caused by a quantum effect.

In section 2, the quantum Gaussian model in quantum estimation, which often appears in quantum optics, is reviewed. The explicit formula of the relative entropy on the quantum Gaussian model is also given. KMB Fisher information is obtained from the formula. In section 3, geometrical quantities such as the Riemannian curvature tensor are calculated based on KMB Fisher information. The scalar curvature on the quantum Gaussian model is shown to be negative. In section 4, the Taylor expansion with respect to the Planck constant $\hbar$ around 0 is proposed. From this expansion, these geometrical quantities are shown to reduce to those defined on the classical model of Gaussian distributions as $\hbar \to 0$. It is further shown that such a method is not applicable to a finite-dimensional system such as the spin system.

2. KMB Fisher information on the quantum Gaussian model

The quantum Gaussian state, which often appears in quantum optics and represents the thermal noise of the quantum harmonic oscillator, is explained in this section. Generally, the quantum Gaussian state has three real parameters and is given by

$$\rho_{\zeta,N} := \frac{1}{\pi N}\int_{\mathbb{C}} \exp\left(-\frac{|\alpha-\zeta|^2}{N}\right)|\alpha\rangle\langle\alpha| \mathrm{d}^2\alpha,$$

where $|\alpha\rangle$ is the coherent vector in the Boson–Fock representation, $N > 0$ is the expectation of the number operator and denotes the average number in the excited mode of the harmonic oscillator and $\zeta = \mathrm{Re}\,\zeta + \mathrm{i}\mathrm{Im}\,\zeta \in \mathbb{C}$ is referred to as the mean parameter.

In thermal equilibrium of temperature T, N is determined as a function of T:

$$N = \left\{\exp\left(\frac{\hbar\omega}{kT}\right) - 1\right\}^{-1},$$

where ω is the frequency of the harmonic oscillator and k is the Boltzmann constant.

The mean parameter ζ represents the mean of the electromagnetic field. Decades ago, optical communication, in which information can be transmitted by adjusting the value of ζ, was considered. As such, the estimation of the mean parameter, including the optimization of measurement devices, was investigated (see, e.g., chapter 6 in Holevo [12]).

Next, a parametric family of the quantum Gaussian states with unknown mean parameter and variance is considered:

$$\mathcal{M} := \{\rho_\theta : \theta^1 = \mathrm{Re}\,\zeta \in \mathbb{R}, \theta^2 = \mathrm{Im}\,\zeta \in \mathbb{R}, \theta^3 = N > 0\}.$$

This model is referred to as the *quantum Gaussian model*. For parameter estimation of the quantum Gaussian model, refer to [8] and the references therein.

Recently, Tanaka and Komaki [28] investigated the Bayesian prediction of the quantum Gaussian model and obtained the following formula of the quantum relative entropy for this model:

$$\begin{aligned} D(\rho_\theta \| \rho_{\theta'}) &= D(\rho_{\zeta,N} \| \rho_{\zeta',N'}) \\ &= \log\left(\frac{N'+1}{N+1}\right) + N\log\left(\frac{N}{N+1}\frac{N'+1}{N'}\right) + \log\left(\frac{N'+1}{N'}\right)|\zeta - \zeta'|^2. \end{aligned} \tag{5}$$

The above formula (5) is the starting point of the argument of the present paper.

From equation (5), KMB Fisher information on the quantum Gaussian model is obtained as follows:

$$\left(g_{ij}^{\mathrm{KMB}}\right) = \begin{pmatrix} 2\log\frac{N+1}{N} & 0 & 0 \\ 0 & 2\log\frac{N+1}{N} & 0 \\ 0 & 0 & \frac{1}{N} - \frac{1}{N+1} \end{pmatrix}. \tag{6}$$

In the following section, geometrical quantities based on the above metric (6) are given.

3. Geometrical structure derived from KMB Fisher information

Once a Riemannian metric is defined on a quantum model manifold, geometrical quantities are derived from the metric. See, e.g., Kobayashi and Nomizu [16] for the differential geometry.

Let us calculate some geometrical quantities. Straightforward calculation yields the Riemannian curvature tensor, as follows:

$$R_{1212} = \frac{1}{(N+1)} - \frac{1}{N},$$

$$R_{1313} = R_{2323} = \frac{1}{2\log\left(\frac{N+1}{N}\right)}\left(\frac{1}{N+1} - \frac{1}{N}\right)^2 + \frac{1}{2}\left\{\frac{1}{(N+1)^2} - \frac{1}{N^2}\right\}.$$

The Ricci tensor and the scalar curvature are also given by

$$R_{11} = R_{22} = -\frac{1}{2}\left(\frac{1}{N+1} + \frac{1}{N}\right),$$

$$R_{33} = 2\left\{\frac{\left(\frac{1}{N+1} - \frac{1}{N}\right)}{2\log\left(\frac{N+1}{N}\right)}\right\}^2 + \frac{\frac{1}{(N+1)^2} - \frac{1}{N^2}}{2\log\frac{N+1}{N}},$$

and

$$R := g^{ij} R_{ji} = 2\frac{\frac{1}{N} - \frac{1}{N+1}}{\left(2\log\frac{N+1}{N}\right)^2} - \frac{\frac{1}{N} + \frac{1}{N+1}}{\log\left(\frac{N+1}{N}\right)}.$$

The scalar curvature is shown to be negative through a lengthy but straightforward calculation. In information geometry, the scalar curvature on the classical Gaussian model is negative constant [1]. Constant curvature reflects the scale invariance of the classical Gaussian model. However, the above curvature is not constant and $R < -\frac{3}{2}$. In the following section, the difference is shown, using the perturbative method, to be due to a quantum effect.

4. Perturbative method

4.1. Simple analysis

As indicated in a standard textbook on quantum mechanics, the classical theory can be seen as an approximation of quantum theory neglecting the Planck scale. If one applies this idea to quantum information geometry, a certain limiting operation may yield the classical counterpart.

Unfortunately, this does not proceed well in the finite-dimensional Hilbert space. Next, the spin $\frac{1}{2}$ system described in a two-dimensional Hilbert space is considered. The noncommutativities in this system are derived from $SU(2)$ algebra. A set of generators in $SU(2)$ is given by a set of traceless Hermitian matrices $\{S_1, S_2, S_3\}$ satisfying

$$[S_i, S_j] = \mathrm{i}\epsilon_{ijk}\frac{\hbar}{2}S_k.$$

Then, noncommutativities vanish when taking the classical limit $\hbar \to 0$. However, since $S_i = \hbar\frac{\sigma_i}{2}$ and the Pauli matrix σ_i is dimensionless, S_i also becomes negligible in the classical limit.

This phenomenon can be easily understood as follows. Since the degree of freedom of spin is purely quantum-mechanical and has no classical counterpart, such a degree of freedom cannot be distinguished in the classical limit.

However, the quantum Gaussian model does not follow the above argument. Taking the classical limit $\hbar \to 0$ (equivalently $[p, q] \to 0$) allows simultaneous projective measurement of both position q and momentum p. Then, when the simultaneous projective measurement is performed, the following classical Gaussian model is obtained:

$$\mathcal{M}_{\mathrm{cl}} := \{p_{\zeta,N}(p, q) : \zeta \in \mathbb{C}, N > 0\},$$
$$p_{\zeta,N}(p, q) := \frac{1}{\pi N}\exp\left\{-\frac{(p - \mathrm{Re}\,\zeta)^2 + (q - \mathrm{Im}\,\zeta)^2}{N}\right\}, \tag{7}$$

where $p_{\zeta,N}(p, q)$ represents the joint distribution of momentum and position [12]. Therefore, the quantum Gaussian model with such a measurement reduces to the classical Gaussian model. Generally, in the canonical conjugate relation (CCR) representation, noncommutativities arise in the Planck scale and the CCR representation has a nontrivial classical counterpart, which is different from finite-dimensional unitary representations, such as the spin representation.

Let us consider the above argument in detail. First, recall that the Hamiltonian of the harmonic oscillator

$$H := \hbar\omega(a^* a) + \tfrac{1}{2}\hbar\omega.$$

For the ground state $|0\rangle$, the expectation value of H is $\langle H\rangle = \langle 0|H|0\rangle = \frac{1}{2}\hbar\omega$. It represents the quantum fluctuation by the zero-point oscillation. On the other hand, for the quantum Gaussian state ρ_θ, we obtain

$$\langle H\rangle = \mathrm{Tr}\,\rho_\theta H = \hbar\omega(N + |\zeta|^2) + \tfrac{1}{2}\hbar\omega.$$

Intuitively speaking, the first term represents the classical energy of the harmonic oscillator. If the quantum fluctuation is much smaller than the classical energy, $\hbar\omega \ll (N + |\zeta|^2)\hbar\omega$ holds. Here, we adopt another parametrization

$$\tilde{N} := \hbar N, \qquad \tilde{\zeta} := \sqrt{\hbar}\zeta.$$

Note that $\hbar$ can be considered as a scaling parameter. In the above parametrization, the quantum Gaussian state is rewritten as

$$\tilde{\rho}_{\tilde{\zeta},\tilde{N}} := \frac{1}{\pi\tilde{N}}\int_{\mathbb{C}} \exp\left(-\frac{|\tilde{\alpha}-\tilde{\zeta}|^2}{\tilde{N}}\right)\left|\frac{\tilde{\alpha}}{\sqrt{\hbar}}\right\rangle\left\langle\frac{\tilde{\alpha}}{\sqrt{\hbar}}\right| \mathrm{d}^2\tilde{\alpha},$$

where $\tilde{\alpha} = \sqrt{\hbar}\alpha$.

We consider the situation where both $\tilde{N}$ and $|\tilde{\zeta}|^2$ are much larger than $\hbar$. For fixed ω, this situation implies that the quantum fluctuation is negligible. Since the inner product of the two coherent vectors $|\tilde{\alpha}/\sqrt{\hbar}\rangle$, $|\tilde{\beta}/\sqrt{\hbar}\rangle$ is given by

$$\left|\left\langle\frac{\tilde{\alpha}}{\sqrt{\hbar}}\middle|\frac{\tilde{\beta}}{\sqrt{\hbar}}\right\rangle\right|^2 = \exp\left\{-\frac{|\tilde{\alpha}-\tilde{\beta}|^2}{\hbar}\right\},$$

the orthogonal relation is obtained when $\hbar \to 0$ for fixed $\tilde{\alpha}, \tilde{\beta}$. The orthogonal relation also implies that the positive operator valued measure (POVM)

$$\left\{\frac{\hbar}{\pi}\left|\frac{\tilde{\alpha}}{\sqrt{\hbar}}\right\rangle\left\langle\frac{\tilde{\alpha}}{\sqrt{\hbar}}\right|\mathrm{d}^2\tilde{\alpha}\right\}_{\tilde{\alpha}\in\mathbb{C}}$$

reduces to the simultaneous projective measurement.

The quantum relative entropy is rewritten as

$$D(\tilde{\rho}_{\tilde{\theta}}\|\tilde{\rho}_{\tilde{\theta}'}) := \log\left(\frac{\tilde{N}'+\hbar}{\tilde{N}+\hbar}\right) + \frac{\tilde{N}}{\hbar}\log\left(\frac{\tilde{N}}{\tilde{N}+\hbar}\frac{\tilde{N}'+\hbar}{\tilde{N}'}\right) + \frac{1}{\hbar}\log\left(\frac{\tilde{N}'+\hbar}{\tilde{N}'}\right)|\tilde{\zeta}-\tilde{\zeta}'|^2. \tag{8}$$

The above quantity and other geometrical quantities derived from it are expanded with respect to $\hbar$ and this is the basic idea of our perturbative method discussed in the next subsection.

Previously, in order to avoid difficulties related to functional analysis, arguments in quantum information geometry were usually restricted to finite-dimensional Hilbert spaces [1]. However, it is important to consider quantum information geometry for models described in infinite-dimensional Hilbert spaces because the quantum Gaussian model can be seen as a quantum analogue of the classical Gaussian model, and the classical Gaussian model is taken as a typical example in classical information geometry. In particular, as long as the CCR representation is used, the Taylor expansion with respect to the Planck constant, which is often used in quantum physics, is also valid.

4.2. Perturbative method

Generally, it is often difficult to calculate geometrical quantities explicitly in a specific model manifold. In classical information geometry, asymptotic expansion (see, e.g., van der Vaart [29]) is used as an approximation method. On the other hand, in quantum information geometry, such kinds of expansion are impossible because the probability spaces change according to measurements [12]. However, the Taylor expansion with respect to $\hbar$ is available, and in the present subsection we see that this expansion is useful to see the contrast between the quantum information geometry and the classical counterpart in the quantum Gaussian model. This method is like a perturbation technique in quantum mechanics and here we call it *perturbative method*.

From now on, we write N instead of $\tilde{N}$ and so on. In equation (8), let us consider the Taylor expansion with respect to $\hbar$. Then, we obtain

$$\begin{aligned} D(\rho_\theta \| \rho_{\theta'}) &= \left\{ \log \frac{N'}{N} + \left(\frac{N}{N'} - 1 \right) + \frac{|\zeta - \zeta'|^2}{N'} \right\} \\ &\quad - \frac{1}{2N} \left\{ \left(\frac{N}{N'} - 1 \right)^2 + \frac{N}{N'} \frac{|\zeta - \zeta'|^2}{N'} \right\} \hbar + O(\hbar^2) \\ &= D(p_\theta \| p_{\theta'}) + O(\hbar). \end{aligned}$$

Note that the principal term is the Kullback–Leibler divergence on the classical Gaussian model (7).

Thus, the parameter $\hbar$ is considered to represent the strength of the noncommutativity. In other words, the relative entropy in the quantum Gaussian model coincides with the classical counterpart in the first order of $\hbar$, and higher order terms can be regarded as a quantum correction if $\hbar$ is relatively small. The Taylor expansion around $\hbar \sim 0$ is referred to as the *perturbative expansion*.

This viewpoint applies also to KMB Fisher information given by (6). Using the Taylor expansion, the following is easily obtained:

$$g_{ij}^{\mathrm{KMB}}(\hbar) = g_{ij}^{(\mathrm{cl})} + g_{ij}^{(1)}\hbar + O(\hbar^2),$$

where

$$g_{ij}^{(\mathrm{cl})} = \begin{pmatrix} \frac{2}{N} & 0 & 0 \\ 0 & \frac{2}{N} & 0 \\ 0 & 0 & \frac{1}{N^2} \end{pmatrix}, \qquad g_{ij}^{(1)} = \begin{pmatrix} -\frac{1}{N^2} & 0 & 0 \\ 0 & -\frac{1}{N^2} & 0 \\ 0 & 0 & -\frac{1}{N^3} \end{pmatrix}.$$

The former quantity is the Fisher information on the classical Gaussian model.

Likewise, the Taylor expansion of other geometrical quantities can be obtained as follows:

$$R_{1313} = R_{2323} = -\frac{1}{2N^3} + \frac{3\hbar}{4N^4} + O(\hbar^2),$$

$$R_{1212} = -\frac{1}{N^2} + \frac{\hbar}{N^3} + O(\hbar^2),$$

and

$$R_{11} = R_{22} = -\frac{1}{N} + \frac{\hbar}{2N^2} + O(\hbar^2),$$

$$R_{33} = -\frac{1}{N^2} + \frac{\hbar}{2N^3} + O(\hbar^2).$$

Finally, the scalar curvature is expanded as

$$R = -\frac{3}{2} + \frac{3}{4}\left(\frac{\hbar}{N}\right) + O(\hbar^2). \tag{9}$$

The following comments pertain to the above-mentioned results. First, the fact that $g^{(\mathrm{cl})} \geqslant g^{\mathrm{KMB}}$, where $A \geqslant B$ denotes that $A - B$ is semi-positive definite, implies that the Gaussian states can be better distinguished if the quantum fluctuation is sufficiently negligible.

Second, R is equal to $R^{(\mathrm{cl})} = -\frac{3}{2}$, the scalar curvature on the classical Gaussian model (7), in the first order of $\hbar$.

Third, KMB Fisher information is compared with that based on RLD and SLD. A number of studies have investigated the question as to which metric is suitable for the space of density matrices [3, 26]. According to Petz [24], the real part of RLD Fisher information is the maximum among the normalized monotone metrics, and SLD Fisher information is the minimum in the space of density operators, as shown below. In the quantum Gaussian model with original parametrization (or taking $\hbar = 1$), RLD Fisher information [7, 12, 30] is given by

$$g^{\mathrm{RLD}} = \begin{pmatrix} \frac{2N+1}{N(N+1)} & \frac{+i}{N(N+1)} & 0 \\ \frac{-i}{N(N+1)} & \frac{2N+1}{N(N+1)} & 0 \\ 0 & 0 & \frac{1}{N(N+1)} \end{pmatrix}$$

and SLD Fisher information [12] is given by

$$g^{\mathrm{SLD}} = \begin{pmatrix} \frac{2}{N+\frac{1}{2}} & 0 & 0 \\ 0 & \frac{2}{N+\frac{1}{2}} & 0 \\ 0 & 0 & \frac{1}{N(N+1)} \end{pmatrix}.$$

Straightforward calculation yields

$$\mathrm{Re}\, g^{\mathrm{RLD}} \geqslant g^{\mathrm{KMB}} \geqslant g^{\mathrm{SLD}}.$$

This is not a surprising result, but rather a consequence of the theorem by Petz. However, it is interesting to observe what happens if the imaginary part of g^{RLD} is included. Though $\mathrm{Tr}\, g^{\mathrm{RLD}} \geqslant \mathrm{Tr}\, g^{\mathrm{KMB}}$,

$$\det\left(g^{\mathrm{RLD}} - g^{\mathrm{KMB}}\right) < 0,$$

for smaller N, where the third component is neglected because g_{i3} is common in the above three metrics. Thus, the relationship $g^{\mathrm{RLD}} \geqslant g^{\mathrm{KMB}}$ does not necessarily hold.

5. Summary

From equation (6), KMB Fisher information was derived and some geometrical quantities on the quantum Gaussian model were calculated. Straightforward calculation shows that the scalar curvature on the quantum Gaussian model is negative but not constant, which implies that the scale invariance is broken. By introducing the Taylor expansion with respect to $\hbar$, the principal terms are equal to those derived from the classical Gaussian model. It is a well-known fact in classical information geometry that the scale invariance holds in the classical Gaussian case and equation (9) indicates that the scale invariance of the Gaussian model is broken in the first order of $\hbar$. Physically, it is possible to interpret the geometry in the classical world as being slightly modified due to a quantum correction. In other words, the smaller the scale of the physical system, the more dominant the quantum fluctuation becomes. This implies a certain mathematical similarity to quantum gravity. We believe that this kind of expansion with respect to $\hbar$ is useful as an approximation method in quantum information geometry while another application is under study.

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References

[1] Amari S and Nagaoka H 2000 *Methods of Information Geometry* (Providence, RI: American Mathematical Society)

[2] Andai A 2003 Monotone Riemannian metrics on density matrices with nonmonotone scalar curvature *J. Math. Phys.* **44** 3675–88

[3] Braunstein S L and Caves C M 1994 Statistical distance and the geometry of quantum states *Phys. Rev. Lett.* **72** 3439–43

[4] Čencov N N 1982 *Statistical Decision Rules and Optimal Inference* (Providence, RI: American Mathematical Society) (Originally published in Russian, (Moscow: Nauka) 1972.)

[5] Dittmann J 2000 On the curvature of monotone metrics and a conjecture concerning the Kubo-mori metric *Linear Algebr. Appl.* **315** 83–112

[6] Hayashi M 2000 Asymptotic quantum theory for the thermal states family *Quantum Communication, Computing and Measurement 2* (Dordrecht/New York: Kluwer/Plenum) pp 99–104

[7] Hayashi M 2002 Two quantum analogues of Fisher information from a large deviation viewpoint of quantum estimation *J. Phys. A: Math. and Gen.* **35** 7689–727

[8] Hayashi M 2005 *Asymptotic Theory of Quantum Statistical Inference* (Singapore: World Scientific)

[9] Helstrom C W 1976 *Quantum Detection Theory* (New York: Academic)

[10] Hiai F and Petz D 1991 The proper formula for relative entropy and its asymptotics in quantum probability *Commun. Math. Phys.* **143** 99–114

[11] Hiai F, Petz D and Toth G 1996 Curvature in the geometry of canonical correlation *Stud. Sci. Math. Hung.* **32** 235–49

[12] Holevo A S 1982 *Probabilistic and Statistical Aspects of Quantum Theory* (Amsterdam: North-Holland)

[13] Holevo A S 1998 The capacity of the quantum channel with general signal states *IEEE Trans. Inform. Theory* **44** 269–73 (*Preprint* quant-ph/9611023)

[14] Janyszek H 1986 On the Riemannian metrical structure in the classical statistical equilibrium thermodynamics *Rep. Math. Phys.* **24** 1–10

[15] Janyszek H 1986 On the geometrical structure of the generalized quantum Gibbs states *Rep. Math. Phys.* **24** 11–9

[16] Kobayashi S and Nomizu K 1969 *Foundations of Differential Geometry II* (New York: Wiley)

[17] Komaki F 2006 Shrinkage priors for Bayesian prediction *Ann. Stat.* (to appear)

[18] Kubo R, Toda M and Hashitsume N 1991 *Statistical Physics II, Nonequilibrium Statistical Mechanics (Springer Series in Solid State Sciences)* (Berlin: Springer)

[19] Michor P, Petz D and Andai A 2000 On the curvature of a certain Riemannian space of matrices *Infinite Dimens. Anal. Quantum Probab. Relat. Top.* **3** 199–212

[20] Nagaoka H 1987 On Fisher information of quantum statistical models. *SITA'87* pp 241–246
Hayashi M (ed) 2005 *Asymptotic Theory of Quantum Statistical Inference* (Singapore: World Scientific) pp 113–24

[21] Ogawa T and Nagaoka H 2000 Strong converse and Stein's lemma in quantum hypothesis testing *IEEE Trans. Inform. Theory* **46** 2428–33

[22] Ohya M and Petz D 2004 *Quantum Entropy and Its Use* (Berlin: Springer)

[23] Petz D 1994 Geometry of canonical correlation on the state space of a quantum system *J. Math. Phys.* **35** 780–95

[24] Petz D 1996 Monotone metrics on matrix spaces *Linear Algebr. Appl.* **244** 81–96

[25] Schumacher B and Westmoreland M D 1997 Sending classical information via noisy quantum channels *Phys. Rev.* A **56** 131–8

[26] Slater P B 1998 Comparative noninformativities of quantum priors based on monotone metrics *Phys. Lett.* A **247** 1–8

[27] Tanaka F and Komaki F 2005 Bayesian predictive density operators for exchangeable quantum-statistical models *Phys. Rev.* A **71** 052323

[28] Tanaka F and Komaki F 2005 Bayesian prediction of the Gaussian states from n sample *Preprint* quant-ph/0510176

[29] van der Vaart A W 1998 *Asymptotic Statistics* (Cambridge: Cambridge University Press)

[30] Yuen H P and Lax M 1973 Multiple-parameter quantum estimation and measurement of nonselfadjoint observables *IEEE Trans. Inform. Theory* **19** 740–50